Equivalent Capacitances for Microstrip Gaps and Steps

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Abstract—**The excess charge density distribution near gaps and steps in microstrip transmission lines is calculated by the solution of singular integral equations. Data are presented for gaps in microstrips of width-to-substrate-thickness ratios of 0.5, 1.0, and 2.0 and relative dielectric constants ranging from 1.0 to 15.0. For steps in a microstrip line results are given for width-to-thickness ratio of unity, relative dielectric constants of 1.0 and 9.6, while the change of width-to-height ratio is from 0.1 to 10.0. The excess charges are calculated explicitly in relatively short computing times, and the results are believed to be accurate to within a few percent.**

Introduction

In RECENT YEARS there has been a growing interest in modeling microstrip discontinuities. Various authors [1]–[5] have obtained results for microstrip open circuits. It appears, however, that few such attempts have been made for the gaps and steps in microstrip, illustrated in Fig. 1. Stinehelfer [1] performed transmission-loss measurements on gaps in microstrip and modeled them by simple gap capacitances as in Fig. 2.

The model proposed here for the gap is a π network, as shown in Fig. 3, while for the step it is a shunt capacitance, as shown in Fig. 4. The mathematical approach taken here is based on the integral equation method used, in [5], to calculate the excess capacitance of a microstrip open circuit.

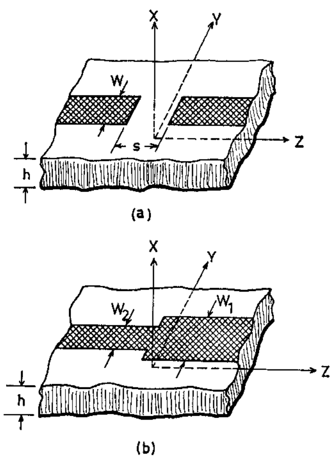

Fig. 1. (a) Gap in a microstrip line. (b) Step in a microstrip line.

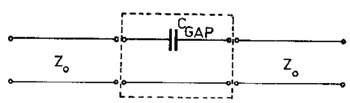

Fig. 2. Stinehelfer's [1] gap capacitance model.

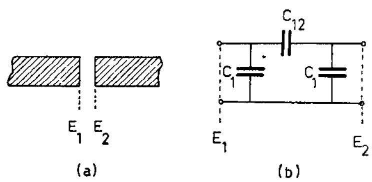

Fig. 3. Capacitive π-network model for a gap in a microstrip line.

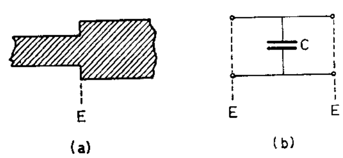

Fig. 4. Shunt capacitance model for a step in a microstrip line.

Formulation of the Gap Problem

The π model in Fig. 3(b) is a symmetric two-port network, so that at least two measurements are required to determine the parameters C_1 and C_{12}. Fig. 5(a) and (b) illustrates the two calculations to be performed. The resulting capacitances are denoted by C_{even} and C_{odd}, respectively.

Let $\phi(P)$ be the potential corresponding to a charge distribution $\sigma(P')$, so that

$$\phi(P) = \int \sigma(P')G(P;P')dP' \tag{1}$$

where $G(P;P')$ is some Green's function appropriate to the particular problem. P and P' are space points for potential and charge, respectively. Let $\phi_\infty(P)$ be the potential (constant on the strip) due to an infinitely extending microstrip line, with corresponding charge density distribution $\sigma_\infty(P')$. Then

$$\phi_\infty(P) = \int \sigma_\infty(P')G_\infty(P;P')dP' \tag{2}$$

where $G_\infty(P;P')$ is the Green's function for the infinite microstrip. Let $\phi_\xi(P)$ be the potential associated with a charge distribution $\sigma_\infty(P')$ for $z>\xi$ and $-\sigma_\infty(P')$ for $z<\xi$. The z-coordinate axis corresponds to the axis of the microstrip, as indicated in Fig. 1. Then

$$\phi_\xi(P) = \int \sigma_\infty(P')G_\xi(P;P')dP' \tag{3}$$

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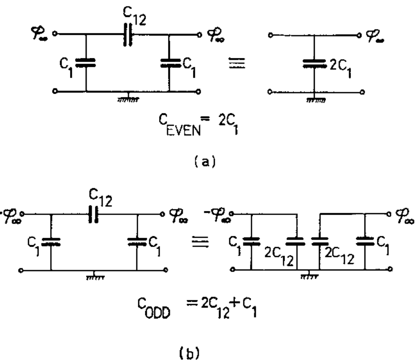

Fig. 5. (a) Symmetrically excited two-port network resulting in C_{even}. (b) Antisymmetrically excited two-port network in C_{odd}.

where $G_\xi(P; P')$ is the Green's function for the charge distribution with polarity reversal at $z=\xi$.

To obtain C_{even}, as defined by Fig. 5(a), an infinitely extending microstrip line is considered, as given by (2). Two other lines with charge density distribution $\frac{1}{2}\sigma_\infty(P')$, each having a polarity reversal at $z=s/2$ and $z=-s/2$, respectively, are governed, according to (3), by

$$\frac{1}{2}\phi_{s/2}(P) = \frac{1}{2}\int \sigma_\infty(P')G_{s/2}(P; P')dP' \tag{4}$$

and

$$\frac{1}{2}\phi_{-s/2}(P) = \frac{1}{2}\int \sigma_\infty(P')G_{-s/2}(P; P')dP'. \tag{5}$$

The superposition of these lines is accomplished by adding (2) and (4), and subtracting (5), thereby resulting in

$$\phi_\infty(P) + \frac{1}{2}\{\phi_{s/2}(P) - \phi_{-s/2}(P)\} = \int \sigma_\infty(P')\Big\{G_\infty(P; P') + \frac{1}{2}[G_{s/2}(P; P') - G_{-s/2}(P; P')]\Big\}dP'. \tag{6}$$

The quantity on the left of (6) represents the potential corresponding to a microstrip charge distribution for $|z|>s/2$ and zero charge elsewhere. Note that on the strips this potential is not ϕ_∞ but rather $\phi_\infty+\frac{1}{2}\{\phi_{s/2}(P) -\phi_{-s/2}(P)\}$.

Now it may be observed that a certain amount of extra charge $\sigma_e^{\text{even}}=\sigma-\sigma_\infty$ must be added to the two strips in Fig. 1(a) to raise the potential on them to ϕ_∞. The potential corresponding to the extra charge is σ_e^{even} is $\phi_\infty-\{\phi_\infty+\frac{1}{2}[\phi_{s/2}(P)-\phi_{-s/2}(P)]\}$, so that

$$\frac{1}{2}[\phi_{-s/2}(P) - \phi_{s/2}(P)] = \int \sigma_e^{\text{even}}(P')G^{\text{even}}(P; P')dP'. \tag{7}$$

Solving (7) for $\sigma_e^{\text{even}}(P')$ gives

$$C_{\text{even}} = \frac{2\int \sigma_e^{\text{even}}(P')dP'}{\phi_\infty} \tag{8}$$

where due to symmetry the integration is performed only over one of the strips.

To evaluate C_{odd}, as defined by Fig. 5(b), it can be shown by an analogous procedure that to raise (lower) the potential on the semi-infinite strip at $z>s/2$ ($z<s/2$), to $\phi_\infty(-\phi_\infty)$ an extra charge $\sigma_e^{\text{odd}}(-\sigma_e^{\text{odd}})$ is required. The corresponding integral equation is

$$\phi_\infty - \frac{1}{2}\{\phi_{s/2}(P) + \phi_{-s/2}(P)\} = \int \sigma_e^{\text{odd}}(P')G^{\text{odd}}(P; P')dP' \tag{9}$$

and C_{odd} is evaluated by

$$C_{\text{odd}} = \frac{\int \sigma_e^{\text{odd}}(P')dP'}{\phi_\infty} \tag{10}$$

where the integration is done over the strip located at $z>s/2$.

Formulation of the Step Problem

The capacitance associated with a sudden change of width of the microstrip line, shown in Fig. 1(b), may be handled in the same way as the C_{even} case was for the gap. Equation (2) is applied to an infinitely extending microstrip line of width-to-height ratio w_1/h, with a charge density distribution of $\frac{1}{2}\sigma_\infty^{(1)}(P')$, so that

$$\frac{1}{2}\phi_\infty^{(1)}(P) = \frac{1}{2}\int \sigma_\infty^{(1)}(P')G_\infty(P; P')dP'. \tag{11}$$

The superscript refers to the width-to-height ratio w_1/h. Similarly, for a microstrip line of w_2/h with a charge density distribution of $\frac{1}{2}\sigma_\infty^{(2)}(P')$

$$\frac{1}{2}\phi_\infty^{(2)}(P) = \frac{1}{2}\int \sigma_\infty^{(2)}(P')G_\infty(P; P')dP'. \tag{12}$$

By (3), for a charge density distribution $\frac{1}{2}\sigma_\infty^{(1)}(P')$ having a polarity reversal at $z=0$, the resulting potential is

$$\frac{1}{2}\phi_0^{(1)}(P) = \frac{1}{2}\int \sigma_\infty^{(1)}(P')G_0(P; P')dP' \tag{13}$$

while for a charge density distribution $\frac{1}{2}\sigma_\infty^{(2)}(P')$ it is

$$\frac{1}{2}\phi_0^{(2)}(P) = \frac{1}{2}\int \sigma_\infty^{(2)}(P')G_0(P; P')dP'. \tag{14}$$

By superposition, adding (11), (12), (13), and subtracting (14), there results

$$\frac{1}{2}\{\phi_\infty^{(1)}(P) + \phi_0^{(1)}(P)\} + \frac{1}{2}\{\phi_\infty^{(2)}(P) - \phi_0^{(2)}(P)\} = \int \sigma_\infty^{(1)}(P')\frac{1}{2}\{G_\infty(P; P') + G_0(P; P')\}dP' + \int \sigma_\infty^{(2)}(P')\frac{1}{2}\{G_\infty(P; P') - G_0(P; P')\}dP'. \tag{15}$$

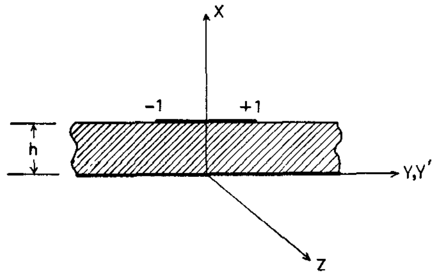

Fig. 6. Cross section of a microstrip line.

The two integrations on the right are carried out over the semi-infinite strips w_1/h, on the interval $z \in (0, \infty)$, and w_2/h, on the interval $z \in (-\infty, 0)$, respectively. Note that the effect is to generate the configuration of Fig. 1(b). Although the charge distribution on the semi-infinite strips is identical to that on microstrip lines of corresponding width-to-height ratios, this is not true of the potentials on the strips.

Let $\sigma_e^{\text{step}}(P')$ be the excess charge density distribution required to produce $\phi_\infty^{(1)}(P)$ and $\phi_\infty^{(2)}(P)$ on the strips. Then the corresponding potentials producing the excess charge are

$$\tfrac{1}{2}\{\phi_\infty^{(1)}(P) - \phi_\infty^{(2)}(P) - \phi_0^{(1)}(P) + \phi_0^{(2)}(P)\}$$

and

$$\tfrac{1}{2}\{\phi_\infty^{(2)}(P) - \phi_\infty^{(1)}(P) - \phi_0^{(1)}(P) + \phi_0^{(2)}(P)\}$$

on the two sides of the plane of discontinuity. Therefore, using (1) for the excess charge, there results

$$\phi^{\text{step}}(P) = \int \sigma_e^{\text{step}}(P')G^{\text{step}}(P; P')dP' \quad (16)$$

where, referring to Fig. 1(b),

$$\phi^{\text{step}}(P) = \frac{1}{2}\begin{cases}\phi_\infty^{(1)}(P) - \phi_\infty^{(2)}(P) - \phi_0^{(1)}(P) + \phi_0^{(2)}(P), & \text{for } z > 0 \\ \phi_\infty^{(2)}(P) - \phi_\infty^{(1)}(P) - \phi_0^{(1)}(P) + \phi_0^{(2)}(P), & \text{for } z < 0.\end{cases} \quad (17)$$

Equation (16) is solved for σ_e^{step}, and then

$$C_{\text{step}} = \frac{\int \sigma_e^{\text{step}}(P')dP'}{\phi_\infty} \quad (18)$$

where the integration is performed over both strips.

Computational Method

The details of the method used to obtain $\sigma_\infty(P')$, the charge distribution of an infinitely extending microstrip transmission line, are as shown in [6]. Briefly, referring to Fig. 6 and using (2), the charge distribution is governed by [6]

$$\phi_\infty(y) = \int_{-1}^{1} \sigma_\infty(y')G_\infty(y; y')dy' \quad (19)$$

where the Green's function, obtained by the method of partial images, is found to be [6]

$$G_\infty(y; y') = \frac{1}{2\pi(\epsilon_0 + \epsilon_1)} \cdot \sum_{n=1}^{\infty} K^{n-1} \log \left\{ \frac{4n^2 + \left(\frac{y - y'}{h}\right)^2}{4(n+1)^2 + \left(\frac{y - y'}{h}\right)^2} \right\} \quad (20)$$

with $K = (\epsilon_0 - \epsilon_1)/(\epsilon_0 + \epsilon_1)$. The potential $\phi_\infty(y)$ is constant on the strip.

The unknown charge distribution $\sigma_\infty(y')$ is even in y' and is expanded in an even set of functions $\{\psi_n\}$ defined by

$$\psi_n(y') = \frac{f_n(y')}{\sqrt{1 - y'^2}} \quad (21)$$

where

$$f_n(y') = \begin{cases} \prod_{i=1}^{n-1} \left\{ \left(\frac{i}{n-1}\right)^2 - y'^2 \right\}, & n > 1 \\ 1, & n = 1. \end{cases} \quad (22)$$

Then the charge density distribution on the strip may be written as

$$\sigma_\infty(y') = \sum a_i \psi_i(y'). \quad (23)$$

Note that the function space $\{\psi_n(y')\}$ contains the expected edge singularity $(1 - y'^2)^{-1/2}$ [7].

When (23) is substituted in (19), it may be solved by projecting both sides on a set of even-order Legendre polynomials. The singularities in the integrand at $y = y'$ and $|y'| = 1$ require special treatment as given in [6].

The Green's function $G_\xi(P; P')$ in (3), obtained by using a line charge with polarity reversal together with partial image theory [5], is

$$G_\xi(h, y, z; y') = \frac{1 - K}{4\pi\epsilon_0}\left\{ f(0) - (1 - K)\sum_{n=1}^{\infty} K^{n-1} f(n) \right\} \quad (24)$$

where

$$f(n) = \log \left\{ \frac{\sqrt{(z-\xi)^2 + 4n^2h^2 + (y-y')^2} + (z - \xi)}{\sqrt{(z-\xi)^2 + 4n^2h^2 + (y-y')^2} - (z - \xi)} \right\}. \quad (25)$$

The charge distribution $\sigma_\infty(y')$, obtained from (19), is used in (3) with the Green's function given in (24), to calculate $\phi_{-s/2}(P)$, $\phi_{s/2}(P)$, $\phi_0^{(1)}(P)$, and $\phi_0^{(2)}(P)$ and, hence, the exciting potential on the left sides of (7), (9), and (17).

The Green's functions $G^{\text{even}}(P; P')$, $G^{\text{odd}}(P; P')$, and $G^{\text{step}}(P; P')$ are obtained, using partial image theory, as described by the authors in [8]. Taking full advantage of the inherent symmetries in Fig. 1(a) and (b) the

Green's functions are found to be

$$G(h, y, z; y', z') = \frac{1}{2\pi(\epsilon_0 + \epsilon_1)h} \cdot \left\{ f(0) - (1 - K) \sum_{n=1}^{\infty} K^{n-1} f(n) \right\} \quad (26)$$

where

$$f^{\text{even}}_{\text{odd}}(n) = \left[(2n)^2 + \left(\frac{y-y'}{h}\right)^2 + \left(\frac{z-z'}{h}\right)^2\right]^{-1/2} + \left[(2n)^2 + \left(\frac{y+y'}{h}\right)^2 + \left(\frac{z-z'}{h}\right)^2\right]^{-1/2} \pm \left[(2n)^2 + \left(\frac{y-y'}{h}\right)^2 + \left(\frac{z+z'}{h}\right)^2\right]^{-1/2} \pm \left[(2n)^2 + \left(\frac{y+y'}{h}\right)^2 + \left(\frac{z+z'}{h}\right)^2\right]^{-1/2} \quad (27)$$

and

$$f^{\text{step}}(n) = \left[(2n)^2 + \left(\frac{y-y'}{h}\right)^2 + \left(\frac{z-z'}{h}\right)^2\right]^{-1/2} + \left[(2n)^2 + \left(\frac{y+y'}{h}\right)^2 + \left(\frac{z-z'}{h}\right)^2\right]^{-1/2}. \quad (28)$$

The excess charge density distribution, $\sigma_e(y', z')$, in either of (7), (9), or (17), is calculated by expanding it in the biquadratic set $\{1, y', z', y'^2, z'^2\}$, and projecting on the set $\{1, y, z, y^2, z^2\}$. The unknown coefficients are obtained by solving the resulting matrix equation [8]. Although it appears that the integrations in the z direction extend to $z = \infty$, in practice the exciting potentials in (7), (9), and (17) fall off to zero rapidly with increasing $|z|$, so that integrations over finite intervals suffice.

Results and Conclusions

C_{even} and C_{odd} normalized to strip width are plotted in Fig. 7 against s/w for substrate relative dielectric constants ranging from 1.0 to 15, and width-to-height ratios of 0.5, 1.0, and 2.0. C_1 and C_{12} to be used in the π model for the gap may be easily calculated using

$$C_1 = \tfrac{1}{2} C_{\text{even}} \quad (29)$$

$$C_{12} = \tfrac{1}{2}[C_{\text{odd}} - C_1]. \quad (30)$$

Equations (29) and (30) follow readily from Fig. 5(a) and (b). As expected, for large values of s/w, $C_{\text{odd}} = C_1$ which in turn approaches the open-circuit capacitance values [6]. Also, as $s/w \rightarrow 0$, C_{even} approaches zero. Transmission-loss calculations indicate that Stinehelfer's [1] gap capacitances for $\epsilon_r = 8.875$ are of the order of 5 percent lower than those presented here for $\epsilon_r = 9.6$.

In Fig. 8 the calculated values of C_{step} are presented for $\epsilon_r = 1.0$ and 9.6, $w_1/h = 1.0$ with $0.1 \leq w_2/h \leq 10.0$. In this case, no published data appear to be available for comparison. It appears that for practical microstrip

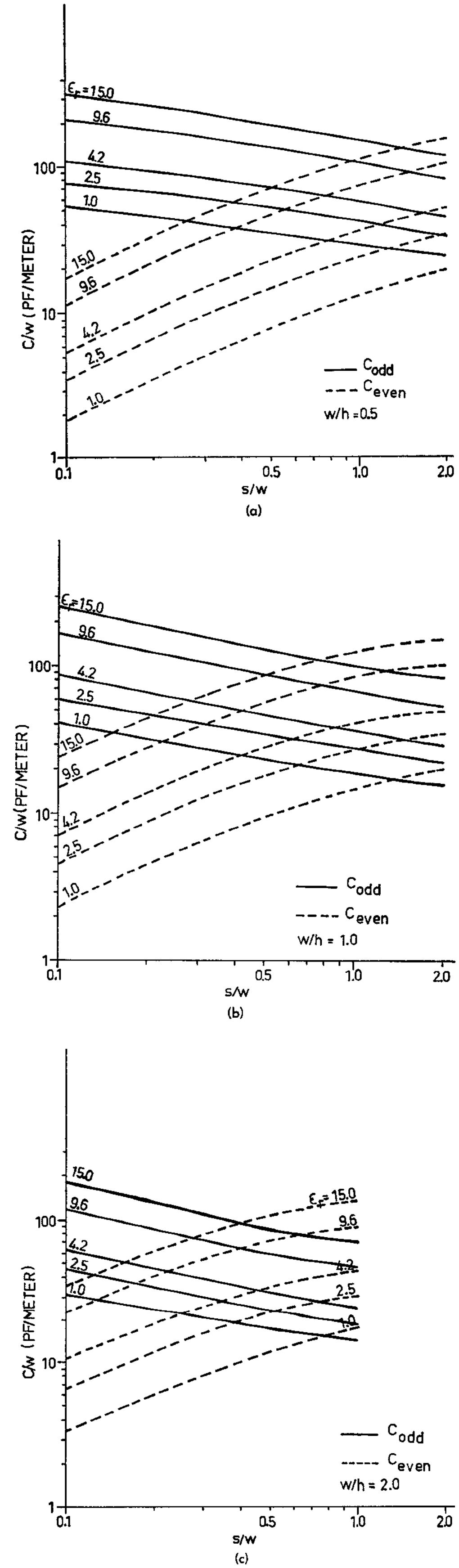

Fig. 7. (a) C_{even} and C_{odd} per unit width of microstrip lines of width-to-height ratio of 0.5 and relative dielectric constants from 1.0 to 15.0. Gap spacing-to-width ratio ranges from 0.1 to 2.0. (b) Same as (a) except width-to-height ratio of 1.0. (c) Same as (a) except width-to-height ratio of 2.0

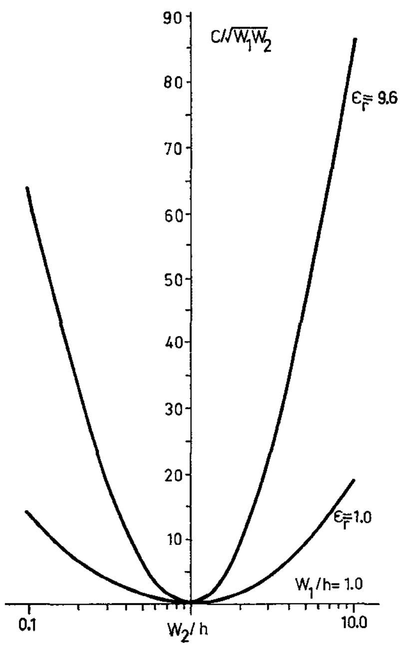

Fig. 8. C_{step} per unit geometric mean width of microstrip lines of width-to-height ratio of unity and relative dielectric constants of 1.0 and 9.6. The change of width-to-height ratio ranges from 0.1 to 10.0.

steps, utilized as quarter-wave transformers, the shunt capacitive contribution is quite small; so that no attempt has been made to produce extensive data.

As the excess charge near gaps and steps in microstrip is calculated explicitly, the problem encountered in the subtraction of nearly equal numbers [4] has been eliminated as it had been in the case of open circuits [5]. The results are believed to be accurate to within a few percent. The calculations were performed on an IBM 360/75 computer. For $\epsilon_r = 9.6$ typical calculation times are about 33 s for C_{even} and C_{odd} and 1 min for C_{step}. For $\epsilon_r = 1.0$ the CPU time required is considerably shorter.

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